On Loops Representing Elements of the Fundamental Group of a Complete Manifold with Nonnegative Ricci Curvature

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Throughout this paper we are concerned with the topological properties of a complete noncompact manifold, M^m. In 1969, Gromoll and Meyer proved that if M^m has positive sectional curvature then it is diffeomeorphic to $\mathbb{R}^m$ [GrMy]. Cheeger and Gromoll proved that if M^m has nonnegative sectional curvature then it is diffeomorphic to the normal bundle over a compact totally geodesic submanifold called the *soul* (1972) [ChGl2]. These theorems do not hold, however, if M^m is only required to have nonnegative Ricci curvature. This is demonstrated by examples of Nabonnand, Wei, and Sha and Yang [Nab],[Wei], [ShaYng]. Nevertheless, there are some serious topological restrictions on such manifolds.

In 1969, Milnor conjectured that a manifold with nonnegative Ricci curvature has a finitely generated fundamental group [Mil]. Thus far there has been significant work done in this area but as yet there is no counter example or proof. Abresch-Gromoll, Anderson, Li, and the author all have partial proofs given additional conditions on volume or diameter [AbrGrl] [Li] [And] [So]. This paper concerns the properties of loops representing elements of the fundamental group for a manifold on which there are no conditions other than those on Ricci curvature. The author hopes it will prove useful to those who are working on this conjecture.

Roughly, we say that M^m has the *loops to infinity property* if given any noncontractible closed curve, C, and any compact set K, there exists a closed curve contained in $M \setminus K$ which is homotopic to C [see Defns 2 and 5]. That is, the loops can "slide" out to infinity beyond any compact set. All the examples mentioned above can be shown to have this property.

In this paper, we prove that if M^m has positive Ricci curvature then it has the loops to infinity property [Thm 8]. As a consequence, it is impossible to take a manifold, remove a ball and edit in a region whose fundamental group has more generators than the fundamental group of the boundary of the removed ball [Thm 17]. Thus one cannot hope to use a sequence of local surgeries to create a manifold with positive Ricci curvature and an infinitely generated fundamental group.

Furthermore, we prove that if a manifold has nonnegative Ricci curvature and does not satisfy the loops to infinity property then M^m is isometric to a flat normal bundle over a compact totally geodesic submanifold [Thm 12]. In fact, M^m has a double cover which splits isometrically [Thm 11].

In the first section of this paper, definitions and theorems are stated precisely with a discussion of examples and consequences. In the second section,

we prove the Line Theorem, Theorem 7, which relates the loops to infinity property to the existence of a line in the universal cover and allows us to apply the Cheeger-Gromoll Splitting Theorem [ChGl1]. In the third section, the Line Theorem and a careful analysis of the deck tranforms, leads to proofs of Theorems 11 and 12.

Local topological consequences of the loops to infinity property and the previous theorems appear in the fourth section. Theorems 17 and Theorem 21, which are partial extensions of theorems of Frankel [Fra], Lawson [Law], and Schoen-Yau [SchYau2], are included in this section. Further applications of these results will appear in work by the author and Zhongmin Shen.

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1 Statements

Let M^m be a complete noncompact manifold. A ray, $\gamma : [0, \infty) \to M$, is a minimal geodesic parametrized by arclength. In contrast, a line, $\gamma : (-\infty, \infty) \to M$ is a minimal geodesic parametrized by arclength in both directions. That is $d(\gamma(t), \gamma(s)) = |t - s|$ for all $s, t \in \mathbb{R}$.

A loop is a closed curve starting and ending at a base point. A geodesic loop is a smooth geodesic except at the base point.

Definition 1 Given a ray γ and a loop $C : [0, L] \to M$ based at $\gamma(0)$, we say that a loop $\bar{C} : [0, L] \to M$ is *homotopic to* C along γ if there exists $r > 0$ with $\bar{C}(0) = \bar{C}(L) = \gamma(r)$ and the loop, constructed by joining γ from 0 to r with C from 0 to L and then with γ from r to 0, is homotopic to C.

Definition 2 An element $g \in \pi_1(M, \gamma(0))$, has the *(geodesic) loops to infinity property along* γ if given any compact set $K \subset M$ there exists a (geodesic) loop, $\bar{C}$, contained in $M \setminus K$ which is homotopic along γ to a representative loop, C, such that $g = [C]$.

It is easy to see that if M is an isometrically split manifold, $M = N \times \mathbf{R}$, with γ in a split direction then every $g \in \pi_1(M, \gamma(0))$ has the geodesic loops to infinity property along γ.

Example 3 If $N^3 = \mathbf{R}^3/G$ where G is the group generated by $g(x, y, z) = (-y, x, z+1)$. Then g has the geodesic loops to infinity property along $(t, 0, 0)$, because line segments in $\mathbf{R}^3$ joining $(t, 0, 0)$ to $(0, t, 1)$ project to geodesic loops. On the other hand g^2 does not have the geodesic loops to infinity property along $(t, 0, 0)$, because line segments in $\mathbf{R}^3$ joining $(t, 0, 0)$ to $(-t, 0, 2)$ all pass through $(0, 0, 1)$. However, it does have the loops to infinity property, because line segments from $(t, 0, 0)$ to $(0, t, 1)$ joined with line segments from $(0, t, 1)$ to $(-t, 0, 2)$ project to the required loops.

Example 4 The complete flat Moebius Strip is $M^2 = \mathbf{R}^2/G$ where G is generated by $g(x, y) = (-x, y+1)$. Note that g doesn't even have the loops to infinity property along $(t, 0) = (-t, 1)$ because curves joining $(t, 0)$ to $(-t, 1)$ must pass through the compact set $(\{0\} \times \mathbf{R})/G$.

Definition 5 M^n has the *(geodesic) loops to infinity* property if given any given any ray, γ, and any element $g \in \pi_1(M, \gamma(0))$, g has the (geodesic) loops to infinity property along γ.

Example 6 Nabannond has constructed an example of a manifold with positive Ricci curvature which is diffeomorphic to $\mathbf{R}^3 \times S^1$ [Nab]. Wei has constructed examples which are diffeomorphic to $\mathbf{R}^k \times N$ where the fundamental group, $\pi_1(N)$, is any torsion free nilpotent group [Wei]. One can show that the examples of Nabonnand and Wei actually satisfy the *geodesic* loops to infinity property.

In Section 2, we will prove the following theorem.

Theorem 7 (Line Theorem) *If M^n is a complete noncompact manifold which does not satisfy the geodesic loops to infinity property then there is a line in its universal cover.*

Recall that the Splitting Theorem of Cheeger and Gromoll states that a manifold with nonnegative Ricci curvature which contains a line splits isometrically [ChGl2] [see also Li's text, Thm 4.2]. Thus we have the following consequence of Theorem 7.

Theorem 8 *If M^n is complete noncompact with Ricci ≥ 0 and there exists $y \in M^n$ such that Ricci$_y > 0$, then M^n has the geodesic loops to infinity property.*

In Section 3, we prove more in the case of nonnegative Ricci curvature.

Proposition 9 If M^n is a complete noncompact manifold with *Ricci* ≥ 0 and there exists an element $g \in \pi_1(M)$ which does not satisfy the *geodesic loops to infinity property* along a given ray γ, then the universal cover splits isometrically, $\tilde{M} = N^{m-k} \times \mathbb{R}^k$. Furthermore the lift $\tilde{\gamma}$ of γ, is in the split direction,

$$\tilde{\gamma}(t) = (x(0), y(t)) \tag{1}$$

and

$$g_*(\tilde{\gamma}'(t)) = -\tilde{\gamma}'(t). \tag{2}$$

Corollary 10 *If M^n is a complete noncompact manifold with Ricci ≥ 0 and $g \in \pi_1(M)$, then either g or g^2 has the geodesic loops to infinity property.*

If we consider manifolds which don't even satisfy the weaker loops to infinity property, we get a stronger result.

Theorem 11 *[Double Cover Theorem] If M^n is a complete noncompact manifold with Ricci ≥ 0 and there exists an element $g \in \pi_1(M)$ which does not satisfy the loops to infinity property along a given ray γ, then all elements $h \in \pi_1(M, \gamma(0))$ satisfy*

$$h_*(\tilde{\gamma}'(t)) = \pm\tilde{\gamma}'(t). \tag{3}$$

Furthermore, M^m has a split double cover which lifts γ to a line.

Local consequences of this theorem appear in Section 4. [Theorem 21].

Cheeger and Gromoll proved that any complete noncompact manifold with nonnegative sectional curvature is diffeomorphic to a normal bundle over a compact totally geodesic submanifold called a soul [ChGr2]. This is not true for manifolds with nonnegative Ricci curvature [ShaYng].

However, M does have a soul if it doesn't have the geodesic loops to infinity property along any ray. This soul is defined using Busemann functions, which are reviewed in Section 2 above Lemma 14.

Theorem 12 *If M^n is a complete noncompact manifold with Ricci ≥ 0 and there exists an element $g \in \pi_1(M)$ which does not satisfy the loops to infinity property along a given ray γ, then the Busemann function, b_γ associated with that ray has a minimum*

$$-s_\gamma = min_{x \in M}(b_\gamma(x))$$

and M^n is a flat normal bundle over $b_\gamma^{-1}(-s_\gamma)$.

If $g \in \pi_1(M)$ doesn't satisfy the geodesic loops to infinity property along any ray, γ, then M^n is a flat normal bundle over a compact totallt geodesic soul, S, where $S = \bigcap_\gamma (b_\gamma^{-1}(-s_\gamma))$.

This is the same soul as the one produced in Cheeger and Gromoll's paper if M^m has nonnegative sectional curvature.

Note that in Example 3, the soul, $S = \{(0,0,z) : z \in [0,1]\}$.

In general, for manifolds with nonnegative Ricci curvature, it is an open question whether, $\bigcap_\gamma (b_\gamma^{-1}(-s_\gamma))$ is compact or not.

In Section 4, we discuss some topological consequences of the loops to infinity property. In particular Theorem 17, states that in a manifold with the loops to infinity property, the group homomorphism induced by the inclusion from $\pi_1(\partial D) \to \pi(Cl(D))$ is a surjection. Thus, if the boundary of a region in a manifold with positive Ricci curvature is simply connected, the region must be simply connected as well. This consequence is an old theorem of Schoen and Yau [SchYau2]. A similar weaker theorem is proven if M^m has $Ricci \geq 0$ [Theorem 21].

2 Proof of the Line Theorem

In this section, M is a Riemannian Manifold and we make no assumptions on Ricci curvature. We begin with a construction of the line in the universal cover. Elements of this proof are used again to prove other theorems in the next section.

Proof of Theorem 7: Let γ be a ray, $g \in \pi_1(M, \gamma(0))$ such that g doesn't satisfy the geodesic loops to infinity property. Let C be a representative of g based at $\gamma(0)$. There exists a compact set K, such that there is no closed geodesic contained in $M \setminus K$ which is homotopic to C along γ. Let $R_0 > 0$ such that $B_{x_0}(R_0) \supset K$.

So for all $r > R_0$, any loop based at $\gamma(r)$ which is homotopic to C along γ must pass through K. Let $r_i > R_0$ be an increasing sequence diverging to infinity.

Let $\tilde{M}$ be the universal cover and $\tilde{C}$ be a lift of C running from $\tilde{x_0}$ to $g\tilde{x_0}$. Since C is noncontractible, g is not the identity and $\tilde{x_0} \neq g\tilde{x_0}$. Let $\tilde{\gamma}$ be the lift of γ starting at $\tilde{x_0}$ and $g\tilde{\gamma}$ be the lift starting at $g\tilde{x_0}$. Then if $\tilde{C_i}$ is a minimal geodesic from $\tilde{\gamma}(r_i)$ to $g\tilde{\gamma}(r_i)$, $C_i = \pi \circ \tilde{C_i}$ is a loop based at $\gamma(r_i)$ which is homotopic to C along γ. Thus there exists t_i such that $C(t_i) \subset K$.

Let $L_i = L(C) = d_{\tilde{M}}(\tilde{\gamma}(r_i), g\tilde{\gamma}(r_i))$.

Let $\tilde{K}$ be the lift of K to the fundamental domain in $\tilde{M}$ such that $\tilde{x_0} \in \tilde{K}$. Note that $\tilde{K}$ is precompact.

For all $i \in \mathbf{N}$ there is an element $g_i \in \pi_1(M, x_0)$ such that $g_i\tilde{C}(t_i) \in \tilde{K}$.

Note that

$$t_i = d_{\tilde{M}}(\tilde{\gamma}(r_i), \tilde{C}(t_i)) \geq d_M(\gamma(r_i), C(t_i)) \geq d_M(\gamma(r_i), K) \geq r_i - R_0$$

and

$$L_i - t_i = d_{\tilde{M}}(g\tilde{\gamma}(r_i), \tilde{C}(t_i)) \geq d_M(\gamma(r_i), C(t_i)) \geq d_M(\gamma(r_i), K) \geq r_i - R_0.$$

So $g_i\tilde{C}$ are minimal geodesics running from $(t_i - (r_i - R_0))$ to $(t_i + (r_i - R_0))$ such that $g_i\tilde{C}(t_i) \in \tilde{K}$. Taking r_i to infinity, a subsequence of $(g_{i*}\tilde{C}'(t_i))$ must converge to a unit vector $(\gamma'_\infty(0))$ based at $\gamma_\infty(0) \in Cl(\tilde{K})$. Furthermore, the geodesic,

$$\gamma_\infty(t) = \exp_{\gamma_\infty(0)}(t\gamma'_\infty(0)) \tag{4}$$

is a line.

$\square$

Note 13 *Note that* $\lim_{i\to\infty} L_i/r_i = 2$ *because* $L_i \leq 2r_i + d(\tilde{\gamma}(0), g\tilde{\gamma}(0))$ *and* $L_i \geq L_i - t_i + t_i \geq 2(r_i - R_0)$.

Recall that given a ray, γ, parametrized by arclength, then the Busemann function associated with that ray, $b_\gamma : M \to \infty$ is defined,

$$b_\gamma(x) = \lim_{R\to\infty} R - d_M(x, \gamma(R)).$$

For example, if M is Euclidean space, then $\gamma(t) = \gamma(0) + t\gamma'(0)$, and

$$b_\gamma(x) = < x - \gamma(0), \gamma'(0) > . \tag{5}$$

If M is a manifold with nonnegative Ricci curvature that contains a line, γ, then by Cheeger and Gromoll, M is the isometric product of $\gamma(\mathbf{R})$ and $b_\gamma^{-1}(\{0\})$.

The following lemma is useful in analyzing the properties of deck transforms in conjunction with rays. It will be used in the next section.

Lemma 14 *If $\tilde\gamma$ is the lift of a ray γ then for all deck transforms g,*

$$b_{\tilde\gamma}(g\tilde\gamma(a)) \le a.$$

Proof: For any $x \in \tilde M$ we have

$$d_{\tilde M}(gx, \tilde\gamma(R)) \ge d_M(\pi(x), \gamma(R)).$$

If we subtract R on both sides and take R to infinity, we get

$$-b_{\tilde\gamma}(gx) \ge -b_\gamma(\pi(x)).$$

Setting $x = \tilde\gamma(a)$, then $\pi(x) = \gamma(a)$ and we are done. $\square$

3 Nonnegative Ricci Curvature

Throughout this section we assume that M^m is a manifold with nonnegative Ricci curvature which does not satisfy the geodesic loops to infinity property. Thus by the Line Theorem, its universal cover, $\tilde M$ contains a line. So by the Splitting Theorem of Cheeger and Gromoll, the universal cover splits isometrically into $N^{m-k} \times \mathbf{R}^k$ where N^{n-k} has no lines and $k \ge 1$.

Let $p_{\mathbf{R}} : \tilde M \to \mathbf{R}^k$ and $p_N : \tilde M \to N$ be the projections. If $g : \tilde M \to \tilde M$ is an isometry, then it acts on each component seperately. [REF Cheeger] That is $g = g|_N : N \to N$ and $g = g|_{\mathbf{R}^k} : \mathbf{R}^k \to \mathbf{R}^k$ are isometries.

The following lemma is quite easy to prove.

Lemma 15 *If $\eta : (a, b) \to \tilde M$ is minimal then $p_{\mathbf{R}} \circ \eta$ and $p_N \circ \eta$ are minimal geodesics between their endpoints. It is possible that one of them is constant.*

Note, however, that the geodesics in this lemma are paremetrized proportional to arclength and are not normalized like the geodesics, rays and lines constructed in the proof of Theorem 7.

We first prove Proposition 9. There is a given ray γ and a given element $g \in \pi_1(M, \gamma(0))$ which does not satisfy the *geodesic loops to infinity property* along γ. We must show that γ lifts to the purely split direction and that $g_*(\tilde{\gamma}'(t)) = -\tilde{\gamma}'(t)$.

Proof of Proposition 9: By the proof of the Line Theorem, we know there are minimal geodesics $\tilde{C}_i$, running from $\tilde{\gamma}(r_i)$ to $g\tilde{\gamma}(r_i)$, whose lengths L_i, are growing like $2r_i$. So, intuitively, $\tilde{\gamma}$ and $g\tilde{\gamma}$ should be in the opposite directions, and thus can only fit in the split direction.

Let $\tilde{C}_i$ be the curves constructed in Theorem 7. Let $x_i(t) = p_N(\tilde{C}_i(t))$ and $y_i(t) = p_{\mathbf{R}}(\tilde{C}_i(t))$. By Lemma 15, these are minimal geodesics from $[0, L_i]$. Since y_i is a minimal geodesic in Euclidean space, it can be written as

$$y_i(t) = y_i'(t_i)(t - t_i) + y_i(t_i)$$

where $t_i \in (0, L_i)$ as in Theorem 7. Since all minimal geods are parametrized proportional to arclength, $|x_i'(t)| = |x_i'(t_i)|$. Since $\tilde{C}_i$ are minimal geodesics parametrized by arclength,

$$|x_i'(t_i)|^2 + |y_i'(t_i)|^2 = |\tilde{C}_i{}'(t_i)|^2 = 1.$$

Now, $g_{i*}(\tilde{C}_i{}'(t_i))$ converges to $\gamma_\infty'(0)$, where γ_∞ is a line defined in (4). Let $x_\infty(t) = p_N(\gamma_\infty(t))$ and $y_\infty(t) = p_{\mathbf{R}}(\gamma_\infty(t))$. By Lemma 15, $x_\infty(t)$ and $y_\infty(t)$ are lines or constants. Since N contains no lines, $x_\infty(t)$ is a constant. Thus

$$|y_\infty'(0)| = |\gamma_\infty'(0)| = 1.$$

Since $g_i|_N$ is an isometry,

$$\lim_{i\to\infty} |x_i'(t_i)| = \lim_{i\to\infty} |g_{i*} x_i'(t_i)| = |x_\infty'(0)| = 0.$$

Similarly,

$$\lim_{i\to\infty} |y_i'(t_i)| = \lim_{i\to\infty} |g_{i*} y_i'(t_i)| = |y_\infty'(0)| = 1. \tag{6}$$

Let $\tilde{\gamma}$ be the lifted ray as in Theorem 7. Let $x = p_N \circ \tilde{\gamma}$ and $y = p_{\mathbf{R}} \circ \tilde{\gamma}$. Recall that $\tilde{C}_i$ is a minimal geodesic of length L_i from $\tilde{\gamma}(r_i)$ to $g\tilde{\gamma}(r_i)$. So x_i is minimal from $x(r_i)$ to $g(x(r_i))$ and y_i is minimal from $y(r_i)$ to $g(y(r_i))$. Thus

$$d_{\mathbf{R}^k}(y(r_i), g(y(r_i))) = L_i |y_i'(t_i)|.$$

Taking $i \to \infty$, and applying (6), we have

$$\lim_{i\to\infty} \frac{d_{\mathbf{R}^k}(y(r_i), g(y(r_i))}{L_i} = 1. \tag{7}$$

Now $y(t)$ is a minimal geodesic in $\mathbf{R}^k$, so $y(t) = y'(0)t + y(0)$ and $g(y(t))$ is also a minimal geodesic, so $g(y(t)) = g_* y'(0)t + g(y(0))$. Thus

$$\begin{aligned} \frac{d_{\mathbf{R}^k}(y(r_i), g(y(r_i))}{L_i} &= \frac{|y'(0)r_i + y(0) - (g_* y'(0)r_i + g(y(0)))|}{L_i} \quad (8) \\ &\le \frac{|y'(0) - g_* y'(0)|r_i + |y(0) - g(y(0)))|}{L_i}. \quad (9) \end{aligned}$$

Putting this together with (7), we have

$$\lim_{i\to\infty} \frac{|y'(0) - g_* y'(0)|r_i + |y(0) - g(y(0)))|}{L_i} = 1. \tag{10}$$

So $|y'(0) - g_* y'(0)| \neq 0$. In particular, $y'(0) \neq 0$, and the original lifted ray $\tilde{\gamma}$ has a component in the split direction.

By Note 13,

$$\lim r_i/L_i = 1/2,$$

so

$$|y'(0) - g_* y'(0)| = 2.$$

However

$$|y'(0)| \le |\tilde{\gamma}'(0)| = 1.$$

Thus

$$2 = |y'(0) - g_* y'(0)| \le |y'(0)| + |g_* y'(0)| = 2|y'(0)| \le 2 \tag{11}$$

and $y'(0) = \tilde{\gamma}'(0)$. So the original lifted ray is completely in the split direction.

Furthermore, to have equalities in (11), the element $g \in \pi_1(M)$ which did not have geodesic loops to infinity must satisy

$$g_* \tilde{\gamma}'(0) = g_* y'(0) = -y'(0) = -\tilde{\gamma}'(0). \tag{12}$$

$\square$

We've completed the proof of Proposition 9 and Corollary 10 follows trivially.

We now turn to a manifold with nonnegative Ricci curvature which does not even have the loops to infinity property. That is, we have a ray γ and an element $g \in \pi_1(M, \gamma(0))$ which does not have the loops to infinity property along γ. Thus, g also does not have the geodesic loops to infinity property.

Proof of Theorem 11: All the conditions of Theorem 9 hold so we know that $\tilde{\gamma}$ lifts to a the split direction of the universal cover and (12) holds.

We first try to construct loops to infinity which may not be geodesic loops. Keep in mind that if M is the moebius strip then there are no such loops while in Example 3, there are.

Now we've shown that $\tilde{\gamma} : [0, \infty) \to \tilde{M}$ lies completely in the split direction, so in fact it extends to a line $\tilde{\gamma} : (-\infty, \infty) \to \tilde{M}$. If this line projects to a line $\gamma = \pi \circ \tilde{\gamma} : (-\infty, \infty) \to M$ then M splits and again we have geodesic loops to infinity. So it does not project to a line.

Suppose there exists $h \in \pi_1(M)$ such that

$$h_*(\tilde{\gamma}'(0)) \neq -\tilde{\gamma}'(0) \text{ and } h_*(\tilde{\gamma}'(0)) \neq \tilde{\gamma}'(0). \tag{13}$$

Then h has geodesic loops to infinity. That is, for all $R > 0$ there exist $r_i > R$ such that the minimal geodesic, η_i, of length, l_i, from $\tilde{\gamma}(r_i)$ to $h\tilde{\gamma}(r_i)$, satisfy

$$\pi(\eta_i([0, l_i])) \cap B_{\gamma(0)}(R) = \emptyset.$$

Furthermore, by the fact that $h_* \in O(n)$ and ((13) holds,

$$(h^{-1}g)_*(\tilde{\gamma}'(0)) = h_*^{-1}(-\tilde{\gamma}'(0)) = -h_*^{-1}(\tilde{\gamma}'(0))/ne - \tilde{\gamma}'(0).$$

Thus $h^{-1}g$ has the geodesic loops to infinity property, and for all $R > 0$ there exist $\bar{r}_i > R$ such that the minimal geodesic, $\bar{\eta}_i$, of length, $\bar{l}_i$, from $\tilde{\gamma}(\bar{r}_i)$ to $h^{-1}g\tilde{\gamma}(\bar{r}_i)$, satisfy

$$\pi(\bar{\eta}_i([0, \bar{l}_i])) \cap B_{\gamma(0)}(R) = \emptyset.$$

Note that $h\bar{\eta}_i$ is a minimal geodesic from $h\tilde{\gamma}(\bar{r}_i)$ to $g\tilde{\gamma}(\bar{r}_i)$ such that

$$\pi(h\bar{\eta}_i([0, \bar{l}_i])) \cap B_{\gamma(0)}(R) = \emptyset.$$

Thus for all $R > 0$ there exists an $r_i > 0$, such that there is a curve, $s : [0, l_i + \bar{l}_i + 2|r_i - \bar{r}_i|] \to \tilde{M}$ running from $\tilde{\gamma}(r_i)$ to $g\tilde{\gamma}(r_i)$. This curve, s, first runs along η_i from $\tilde{\gamma}(r_i)$ to $h\gamma(r_i)$, then along $h\tilde{\gamma}(t)$ to $h\tilde{\gamma}(\bar{r}_i)$, then along

$h\bar{\eta}_i$ to $g\tilde{\gamma}(\bar{r}_i)$, and finally along $g\tilde{\gamma}(t)$ to $g\tilde{\gamma}(r_i)$. Clearly $s(t) = \pi(\tilde{s}(t))$ avoids $B_{\gamma(0)}(R)$ and is homotopic along γ to any curve based at $\gamma(0)$ representing g.

This contradicts the hypthesis in (13), so we have proven (3).

We now construct the double cover. Let

$$H = \{h \in \pi_1(M, \gamma(0)) : h_*(\tilde{\gamma}'(0)) = \tilde{\gamma}'(0)\}.$$

Then H is clearly a normal subgroup of $\pi_1(M, \gamma(0))$ and $\pi_1(M, \gamma(0))/H = \{[e], [g]\}$. Thus there exists a double cover $\bar{M} = \tilde{M}/H$ of M.

Let $\pi_H : \tilde{M} \to \bar{M}$. We claim that $\pi_H(\tilde{\gamma})$ is a line.

Suppose not. Then there exists $s > 0$ such that

$$d_{\bar{M}}(\pi_H(\tilde{\gamma})(-s), \pi_H(\tilde{\gamma})(s)) < 2s.$$

So there exists $\bar{h} \in H$, such that

$$d_{\tilde{M}}(\bar{h}(\tilde{\gamma}(-s)), \tilde{\gamma}(s)) < 2s. \tag{14}$$

Let $(\bar{x_0}, \bar{y_0}) = \bar{h}(\tilde{\gamma}(0))$. Now by Lemma 14,

$$b_{\tilde{\gamma}}((\bar{x_0}, \bar{y_0})) \leq b_{\tilde{\gamma}}(\gamma(0)). \tag{15}$$

Since $\tilde{M}$ is split and $\tilde{\gamma}$ is in the split direction,

$$b_{\tilde{\gamma}}(x_0, y(0)) = b_{\tilde{\gamma}}(x(0), y(0))$$

and as in (5),

$$b_{\tilde{\gamma}}((x(0), y(0) + v)) = b_{\tilde{\gamma}}((x(0), y(0))) + < v, y'(0) >_{\mathbf{R}^k} . \tag{16}$$

Setting $v = \bar{y}(0) - y(0)$ and applying (15), we have

$$\begin{aligned} < \bar{y_0} - y(0), y'(0) > &= b_{\tilde{\gamma}}(x(0), \bar{y_0}) - b_{\tilde{\gamma}}(x(0), y(0)) && (17) \\ &= b_{\tilde{\gamma}}(\bar{x}_0, \bar{y_0}) - b_{\tilde{\gamma}}(x(0), y(0)) \leq 0 && (18) \\ & && (19) \end{aligned}$$

Now $h_*(\tilde{\gamma}'(0)) = \tilde{\gamma}'(0)$, so $h(\tilde{\gamma}(t)) = (\bar{x_0}, -y'(0)t + \bar{y_0})$ while $\tilde{\gamma}(t) = (x(0), -y'(0)t + y(0))$. Thus, by (17),

$$\begin{aligned} d_{\tilde{M}}(\bar{h}(\tilde{\gamma}(-s)), \tilde{\gamma}(s))^2 &\geq d_{\mathbf{R}^k}(-y'(0)(-s) + \bar{y_0}, -y'(0)s + y(0))^2 \\ &= |2sy'(0) + \bar{y_0} - y(0)|^2 \\ &= |2sy'(0)|^2 + 4s < \bar{y_0} - y(0), y'(0) > + |\bar{y_0} - y(0)|^2 \\ &\geq |2sy'(0)|^2. \end{aligned}$$

This contradicts (14), so $\pi_H(\tilde{\gamma})$ is a line in the double cover.

□

We now prove Theorem 12, in which we study the Busemann functions on manifolds which don't satisfy the loops to infinity property.

Proof of Theorem 12:

Now $\gamma(t)$ is not a line, else we would have had geodesic rays to infinity. However, it is possible that $\gamma : [-s, \infty) \to M$ is a ray for some $-s < 0$. Let $s_\gamma > 0$, be defined such that $\gamma : [-s_\gamma, \infty) \to M$ is a ray and $\gamma : [-s, \infty) \to M$ is not a ray for any $s > s_\gamma$.

We claim that any element $h \in \pi_1(M, \gamma(0))$ maps the level set

$$b_{\tilde{\gamma}}^{-1}(-s_\gamma) = N^{n-k} \times b_y^{-1}(-s_\gamma) \subset \tilde{M}$$

to itself. Recall that $\tilde{\gamma}(t) = (x(0), y(t))$ by Theorem 9.

Recall that by (5) and the splitting, if $(z, w) \in \tilde{M}$, then

$$\begin{aligned} b_{\tilde{\gamma}}((z,w)) &= b_y(w) =< w - y(0), y'(0) > && (20)\\ &= < w - y(-s_\gamma), y'(0) > + < y(-s_\gamma) - y(0), y'(0) > && (21)\\ &= < w - y(-s_\gamma), y'(0) > -s_\gamma. && (22)\end{aligned}$$

Suppose $(z_0, w_0) \in b_{\tilde{\gamma}}^{-1}(-s_\gamma)$. Since h preserves the splitting, and satisfies (3),

$$\begin{aligned} b_{\tilde{\gamma}}(h(z_0, w_0)) &= b_{\tilde{\gamma}}((hz_0, hw_0))\\ &= < hw_0 - y(-s_\gamma), y'(0) > -s_\gamma\\ &= < w_0 - h^{-1}y(-s_\gamma), h_*^{-1}y'(0) > -s_\gamma\\ &= \pm < w_0 - h^{-1}y(-s_\gamma), y'(0) > -s_\gamma.\end{aligned}$$

Thus $h(z_0, w_0) \in b_{\tilde{\gamma}}^{-1}(-s_\gamma)$ iff

$$< h^{-1}y(-s_\gamma) - y(-s_\gamma), y'(0) >= 0.$$

So we need only show that $h(\tilde{\gamma}(-s_\gamma)) \in b_{\tilde{\gamma}}^{-1}(-s_\gamma)$.

Since $\tilde{\gamma}$ is a lift of a ray, we can apply Lemma 14. Thus

$$h(\tilde{\gamma}(-s_\gamma)) \in b_{\tilde{\gamma}}^{-1}((-\infty, -s_\gamma]).$$

Now $\gamma : [-s, \infty) \to M$ is not a ray for if $s > s_\gamma$. So there exists $s_i \to s_\gamma$ and $r_i \to \infty$ such that

$$\begin{aligned} r_i + s_i &> d_M(\gamma(-s_i), \gamma(r_i)) && (23)\\ &\geq d_{\tilde{M}}(h\tilde{\gamma}(-s_i), \tilde{\gamma}(r_i)). && (24)\end{aligned}$$

Subtracting r_i and taking a limit as r_i approaches infinity,

$$\begin{aligned} s_\gamma = \lim_{i\to\infty} s_i &\geq \lim_{i\to\infty} (d_M(\gamma(-s_i), \gamma(r_i)) - r_i) && (25)\\ &\geq -\lim_{i\to\infty} (r_i - d_{\tilde{M}}(h\tilde{\gamma}(-s_i), \tilde{\gamma}(r_i)) && (26)\\ &= -\lim_{i\to\infty} (r_i - d_{\tilde{M}}(h\tilde{\gamma}(-s_\gamma), \tilde{\gamma}(r_i)) && (27)\\ &= -b_{\tilde{\gamma}}(h\tilde{\gamma}(-s_\gamma)). && (28) \end{aligned}$$

Thus

$$h(\tilde{\gamma}(-s_\gamma)) \in b_{\tilde{\gamma}}^{-1}([-s_\gamma, \infty),$$

and the claim is proven.

Thus $\tilde{M}$ is a flat normal bundle over $b_{\tilde{\gamma}}^{-1}(-s_\gamma)$ with one dimensional fibres, and $\pi_1(M, \gamma(0))$ is a group which preserves the base and maps fibres to fibres. Thus M is a flat normal bundle over $\pi(b_{\tilde{\gamma}}^{-1}(-s_\gamma))$.

Furthermore $g\tilde{\gamma} \in b_{\tilde{\gamma}}^{-1}(s_\gamma)$ and $g\tilde{\gamma}'(0) = -\tilde{\gamma}'(0)$ implies that

$$b_{g\tilde{\gamma}}(\tilde{p}) - s_\gamma = -(b_{\tilde{\gamma}}(\tilde{p}) - s_\gamma) \qquad \forall \tilde{p} \in \tilde{M}.$$

Thus for any $p \in M$, with lift $\tilde{p}$,

$$\begin{aligned} b_\gamma(p) &= \lim_{R\to\infty} R - d_M(p, \gamma(R))\\ &\geq \lim_{R\to\infty} R - max\{d_{\tilde{M}}(\tilde{p}, \tilde{\gamma}(R)).d_{\tilde{M}}(g\tilde{p}, \tilde{\gamma}(R))\}\\ &\geq min\{b_{\tilde{\gamma}}(\tilde{p}), b_{g\tilde{\gamma}}(\tilde{p})\}\\ &\geq -s_\gamma, \end{aligned}$$

Thus $-s_\gamma = \min_{p\in M} b_\gamma(p)$.

Now suppose there is an element $g \in \pi_1(M, x_0)$, such that for any ray γ with $\gamma(0) = x_0$, g doesn't have the geodesic loop to infinity property. Then for each γ, we have s_γ and splitting in the γ direction such that $\tilde{M}$ is a flat normal bundle over a totally geodesic set $b_{\tilde{\gamma}}^{-1}(-s_\gamma)$ and $\pi_1(M, \gamma(0))$ is a group which preserves the base and maps fibres to fibres.

Thus $\bigcap_\gamma b_{\tilde{\gamma}}-1(-s_\gamma))$ is totally geodesic and

$$\tilde{M} = \bigcap_\gamma b_{\tilde{\gamma}}^{-1}(-s_\gamma)) \times \mathbf{R}^l$$

where

$$l = dim(span\{\gamma'(0) : \gamma \text{ is a ray based at } x_0\}.$$

Thus M is a flat normal bundle over the totally geodesic $S = \bigcap_\gamma b_\gamma^{-1}(-s_\gamma))$.

Now S is totally geodesic, so if it were noncompact it would contain a ray γ. However, no ray is ever contained in its own level set, so no ray can be contained in S. $\square$

4 Topological Consequences of Loops to Infinity

The simplest consequence of the loops to infinity property is the following simple theorem.

Theorem 16 *If M^n has the loops to infinity property, K is a compact set and y_0 is a point in an unbounded component $U \subset M \setminus K$, then the inclusion map*

$$i_* : \pi_1(U, y_0) \longmapsto \pi_1(M, y_0)$$

is onto.

Proof: Since U is unbounded, there exists $R_0 > 0$ and a ray, γ, from y_0 such that $\gamma(r) \in U$ for all $r \geq R_0$.

Given $g \in \pi_1(M, y_0)$, M has the loops to infinity property, so there exists a loop $\bar{C}$ contained in $M \setminus K$ which is homotopic along γ to a representative loop C such that $[C] = g$. Since U is a connected component, $\bar{C} \in U$. Now we can add segments of γ to the front and back of $\bar{C}$ to get a curve η which is homotopic to C, based at y_0 and still contained in U. The $[\eta] \in \pi_1(U, y_0)$ and $i_*([\eta]) = g$. $\square$

The following theorem is a localization of the above and is proven below the statements of its corollaries.

Theorem 17 *Let M^n be a complete riemannian manifold with the loops to infinity property along some ray, γ. Let $D \subset M$ be a precompact region with smooth boundary containing $\gamma(t_0)$ and S be any connected component of ∂D containing a point $\gamma(t_1)$. Then the inclusion map*

$$i_* : \pi_1(S, \gamma(t_1)) \longmapsto \pi_1(Cl(D), \gamma(t_1))$$

is onto.

Note that this theorem is closely related to results of Frankel, Lawson and Schoen-Yau which concern precompact regions in a complete manifold, M^m, with nonnegative Ricci curvature but without the assumption that M^m is noncompact. Frankel and Lawson are able to prove that $i*$ is surjective, but they require that the boundary have conditions on its mean curvature [Fra] [Law]. Schoen and Yau do not require any extra boundary conditions but they have a weaker conclusion than the one in Theorem 17 [SchYau2]. The methods used to prove the above theorems involve Synge's second variation of arclength in [Fra] and [Law], and harmonic maps in [SchYau2].

Corollary 18 *Let M^n has the loops to infinity property. If ∂D is simply connected, then $\pi_1(D)$ is trivial.*

Corollary 19 *If M^n has nonnegative Ricci curvature and ∂D is simply connected, then $\pi_1(D)$ can only contain elements of order 2.*

Corollary 20 *Any Riemannian manifold M^n with the loops to infinity property which is simply connected at infinity, is simply connected.*

Proof of Theorem 17: Since S is smooth and compact, there exists

$$r_0 = \min_{x \in \partial D} \; injrad(x) \; > \; 0,$$

such that the tubular neighborhood $T_{r_0}(S)$ is homotopic to S. The exponential map along the normals can be used to create the homotopy.

Let $U = D \cup T_{r_0}(S)$ and $V = (M \;\setminus D) \cup T_{r_0}(S)$. Then $U \cap V = T_{r_0}(S)$. Note that U is homotopic to D. We wish to prove that:

$$i : \pi(T_{r_0}(S), \gamma(t_0)) \longmapsto \pi(U, \gamma(t_0))$$

is onto. That is, we must show that given any loop, $C_1 \in U$, based at $\gamma(t_0)$ which is not contractible in U, there exists a curve $C_2 \in U \cap V$ based at the same point, which is homotopic to C_1.

Let

$$t_2 = \sup\{t \;s.t.\; \gamma(t) \in D\} \tag{29}$$

Fix $C_1 \in U$ as above. If C_1 is not contractible in M then by the loops to infinity property and the compactness of $Cl(U)$, there exists a loop $C_3 \in$

$M \setminus Cl(U)$ based at some point $\gamma(t_3)$ which is homotopic along $\gamma([t_2,t_3])$ to C_1. If C_1 is contractible in M, then the same statement is true with C_3 equal to a constant curve.

Look at the universal cover $\tilde{M}$. Let $\tilde{U}=\pi^{-1}(U)$ and $\tilde{V}=\pi^{-1}(V)$. Let $\tilde{\gamma}$ be a lift of the ray γ and $g\in\pi(M)$ be the deck transform represented by $[C_1]$. It may be the identity. Let $\tilde{C_1}\in\tilde{U}$ be the lift of C_1 running from $\gamma(\tilde{t_2})$ to $g\gamma(\tilde{t_2})$ and $\tilde{C_3}\in\tilde{M}\setminus\tilde{U}$ be the lift of C_3 running from $\gamma(\tilde{t_3})$ to $g\gamma(\tilde{t_3})$.

Then there exists $H:[0,1]\times[t_2,t_3]\longmapsto\tilde{M}$ such that $H(s,0)=\tilde{C_2}(s)$, $H(s,1)=\tilde{C_3}(s)$, and $H(0,t)=\tilde{\gamma}(t)$ and $H(1,t)=g\gamma(\tilde{t})$. Here we may have to reparametrize C_2 and C_3.

Note that $H^{-1}(\tilde{U})$ and $H^{-1}(\tilde{V})$ are relatively open in $[0,1]\times[t_2,t_3]$ and their union is $[0,1]\times[t_2,t_3]$. We would like to find a curve

$$(s(r),t(r))\subset H^{-1}(\tilde{U})\cap H^{-1}(\tilde{V})\subset[0,1]\times[t_2,t_3] \tag{30}$$

such that $h(0)=(0,t_2)$ and $h(1)=(1,t_2)$. Then

$$C_2(r):=\pi(H(s(r),t(r)))\subset U\cap V$$

is homotopic to C_1 based at $\gamma(t_2)$ and we are done.

To prove this we need only find a connected relatively open set contained in $H^{-1}(\tilde{U})\cap H^{-1}(\tilde{V})$ which contains both $(0,t_2)$ and $(1,t_2)$. This is true because connected open sets in Euclidean space are pathwise connected.

We employ the following lemma from Munkrees textbook [Mnk].

Lemma 13.1 of Munkrees: *Let $W=X\cup Y$ where X and Y are open sets and let $X\cap Y=A\cup B$ where A and B are disjoint open sets. If there exist two paths connecting $a\in A$ to $b\in B$, one contained in X and the other contained in Y then $\pi_1(W,a)\neq 0$.*

We let $W=[0,1]\times[t_2,t_3]$, $X=H^{-1}(\tilde{U})$ and $Y=H^{-1}(\tilde{V})$. Let A be the connected component of $X\cap Y$ which contains $a=(0,t_2)$ and let B be the connected component of $X\cap Y$ which contains $b=(1,t_2)$. I claim $A=B$. If not, they are disjoint and we can apply the lemma. There exists a curve, namely (s,t_2) contained in X joining a to b. By our choice of t_2 in (29), there exists a curve running around the other three sides of the square joining a to b which is contained in Y. So $\pi_1(W,a)\neq 0$. This contradicts the fact that W is contractible. Thus $A=B$ is a connected component of $H^{-1}(U)\cap H^{-1}(V)$ containing both $(0,t_2)$ and $(1,t_2)$.

□

Theorem 21 *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature. Let $D \subset M$ be a precompact region with smooth boundary, γ a ray starting at $\gamma(0) \in D$ and S be any connected component of ∂D containing a point $\gamma(t_1)$. Then the image of the inclusion map*

$$i_* : \pi(S, \gamma(t_1)) \longmapsto \pi(Cl(D), \gamma(t_1))$$

is $N \subset \pi(Cl(D), \gamma(t_1))$ such that $\pi(Cl(D), \gamma(t_1))/N$ contains at most two elements.

In fact, it contains only one element unless u

Proof: If M^n has the loops to infinity property, then by Theorem 17, we know that $\pi(Cl(D), \gamma(t_1))/N = \{e\}$. If it does not, then by Theorem 11, there is a double cover, $\bar{M}$, of M^m, which splits along the lift, $\tilde{\gamma}$ of the geodesic, γ, and has the loops to infinty property.

If $\pi^{-1}(Cl(D))$ is not connected, then $Cl(D)$ is homeomorphic to one of the connected components of its lift. So we can apply Theorem 17 to the connected component, and we see that i_* is a surjection.

If $\pi^{-1}(Cl(D))$ is connected, then it is the double cover of $Cl(D)$. So there exists an element $g \in \pi_1(Cl(D), \gamma(t_1))$ whose representatives are lifted to non-closed paths in $\pi^{-1}(Cl(D))$. We need only show that if $h \in \pi_1(Cl(D), \gamma(t_1))$, then there exists $\bar{h} \in \pi(S, \gamma(t_1))$ such that either $i_*(\bar{h}) = h$ or $i_*(\bar{h}) = gh$.

If $h \in \pi_1(Cl(D), \gamma(t_1))$ then either a representative lifts to closed loops based at $\tilde{\gamma}(t_1))$ in $\pi^{-1}(Cl(D))$ or the representatives of gh do. Let $\tilde{C}$ be the lifted loop.

By the loops to infinity property on the double cover and Theorem 17, there exists an element $\tilde{h} \in \pi_1(\pi^{-1}(S), \tilde{\gamma}(t_1))$ such that $[\tilde{C}] \in \tilde{h}$. Let $\bar{h} = \pi_*(\tilde{h})$. Then $\bar{h} \in \pi_1(S, \gamma(t_1))$ and

$$\bar{h} = [\pi \circ \tilde{C}] = h \text{ or } gh.$$

□

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